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A NUMERICAL METHOD FOR HEAT TRANSFER IN AN EXPANDING ROD

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NOMENCLATURE

c, heat capacity of the rod;
$f(x,t)$, distributed source or sink;
$g[u,u_x,L(t)]$, boundary value defined by equation (3b);
k, thermal conductivity of the rod;
$L(t)$, free end-coordinate of the rod;
t, time variable;
$u(x,t)$, temperature;
$u_n(x)$, approximate temperature at time $t_n = n\Delta t$;
$v_n(x)$, $ku_n'(x)$;
$w_i(x)$, solution of the equation (10);
x, length coordinate;
z_n, auxiliary function derived from equation (7).

Greek symbols

α, coefficient of linear expansion of the rod;
$\alpha(t), \beta(t)$, end temperatures;
$\gamma\delta$, heat-transfer coefficients;
Δt, time step;
Δx, space step;
Δu, temperature drop;
ρ, density of the rod;
ϕ_n, auxiliary function defined by equation (11).

Subscripts

i, 1, 2;
n, 0, 1, ...; indicates time level $t_n = n\Delta t$;
x, denotes partial derivative with respect to x.

Superscript

$'$ d/dx.

INTRODUCTION

IT IS the purpose of this note to describe a simple and reasonably fast numerical technique for computing conductive heat transfer in an expanding rod. An algorithm which specifically accounts for changes in length has several possible uses. It can be employed for the determination of the temperature in a rod or slab where an initially perfect thermal contact is lost because of shrinkage so that the heat flow across the regressing face and the subsequent cooling of the rod are retarded. It can be used to compute transient thermal stresses in rods with significant axial temperature variations, and it may allow a computer matching of thermal expansion data obtained from the heating and cooling periods of push rod dilatometer measurements which currently use only equilibrium temperatures [2].

THE MODEL

The interdependence between the length of the rod and its temperature can be modeled by a free boundary problem. Let us suppose a rod of initial length $L(0)$ is heated (or cooled). If its length at a future time t is $L(t)$ then the temperature distribution in the rod is given by the conduction equation

$$(ku_x)_x - c\rho u_t = f(x,t),\quad 0 < x < L(t),\ t > 0 \tag{1}$$

where $f(x,t)$ accounts for possible sources and sinks, and where k, c and ρ may be space and time dependent. We shall assume that the rod has a known initial temperature distribution

$$u(x,0) = u_0(x),\quad 0 \leqslant x \leqslant L(0). \tag{2}$$

and that at the fixed end the temperature is given as

$$u(0,t) = \alpha(t),\quad t > 0. \tag{3a}$$

The boundary condition on the free end will be written as

$$g[u,u_x,L(t)] = 0. \tag{3b}$$

For example, if the temperature is specified at $L(t)$ we have $g[u,u_x,L(t)] \equiv u[L(t),t] - \beta(t) = 0$ where $\beta(t)$ is a given function. If radiation between a fixed boundary at $L(0)$ with temperature $\beta(t)$ and the regressing boundary at $L(t)$ across the gap $L(0)$–$L(t)$ occurs, we can write, under certain assumptions ([5], p. 324),

$$u_x[L(t),t] = \gamma\{\beta^4(t) - u^4[L(t),t]\}/\{\tfrac{3}{4}[L(0)-L(t)]+\delta\}.$$

As we shall see the specific form of g in equation (3b) does not greatly influence the algorithm.

In order to find the length $L(t)$ we suppose that over a time span of length Δt a small rod segment of length $\Delta x(t)$ is transformed into an expanded (or contracted) segment $\Delta x(t+\Delta t)$ according to the usual linear law

$$\Delta x(t+\Delta t) = (1+\alpha\Delta u)\Delta x(t)$$

where α is the coefficient of expansion and Δu the average temperature change in the segment. Dividing by Δt and letting $\Delta t \to 0$ we find that

$$\frac{\partial}{\partial t}\Delta x(t) = \alpha \frac{\partial u}{\partial t}\Delta x.$$

Summing all the segments and letting $\Delta x \to 0$ we obtain the defining equation for $L(t)$

$$\frac{\partial L(t)}{\partial t} = \int_0^{L(t)} \alpha \frac{\partial u}{\partial t}(x,t)\,dx. \tag{4}$$

The equations (1)–(4) constitute a free boundary problem. They are similar to the heat-transfer equations for fluidized-bed coating which describe a generalized Stefan problem (see e.g. [1, 4]).

THE NUMERICAL METHOD

A fully implicit solution technique based on the method of invariant imbedding will be used. Let us discretize time and replace all time derivatives by backward difference quotients. Thus, heat conduction in an expanding rod at time $t = t_n$ is approximated by

$$(ku_n')' - c\rho\frac{u_n - u_{n-1}(x)}{\Delta t} = f(x, t_n) \tag{5}$$

$$u_n(0) = \alpha(t_n), \quad g(u_n, u_n', L_n) = 0 \tag{6}$$

$$L_n = L_{n-1} + \int_0^{L_n} \alpha[u_n - u_{n-1}(x)]\,dx \tag{7}$$

where u_n and L_n denote the temperature and rod length at time t_n and the primes denote differentiation with respect to x. As is common in the numerical solution of boundary value problems with integral side conditions, the relation (7) is expressed in differential form. To this end let us define the function

$$z_n(x) = \int_0^x \alpha[u_n - u_{n-1}(x)]\,dx$$

so that $z_n(0) = 0$, $z_n(L_n) = L_n - L_{n-1}$ and

$$z_n' = \alpha[u_n - u_{n-1}(x)].$$

If we also set $ku_n' = v_n$ then the equations (5)–(7) are equivalent to the first order system

$$u_n' = \frac{1}{k}v_n, \quad u_n(0) = \alpha(t_n)$$

$$z_n' = \alpha u_n - \alpha u_{n-1}(x), \quad z_n(0) = 0$$

$$v_n' = \frac{c\rho}{\Delta t}[u_n - u_{n-1}(x)] + f(x, t_n),$$

with the additional boundary conditions

$$z_n(L_n) = L_n - L_{n-1}, \quad g(u_n, u_n', L_n) = 0.$$

To this inhomogeneous system of three linear differential equations and four boundary conditions on the unknown interval $[0, L_n]$ we apply the method of invariant imbedding. Using the notation of ([3], p. 68) we shall write

$$\begin{pmatrix} u_n \\ z_n \end{pmatrix} = \mathbf{U}(x)v_n + \mathbf{w}(x) \tag{8}$$

where the vectors

$$\mathbf{U}(x) = \begin{pmatrix} U_1(x) \\ U_2(x) \end{pmatrix} \text{ and } \mathbf{w}(x) = \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}$$

satisfy the initial value problems

$$\mathbf{U}' = \begin{pmatrix} 1/k \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\mathbf{U} - \mathbf{U}\left(\frac{c\rho}{\Delta t}, 0\right)\mathbf{U}, \quad \mathbf{U}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

$$\mathbf{w}' = \left[\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} - \mathbf{U}\left(\frac{c\rho}{\Delta t}, 0\right)\right]\mathbf{w} + \mathbf{U}\left[\frac{c\rho}{\Delta t}u_{n-1}(x) - f(x, t_n)\right] - \begin{pmatrix} 0 \\ \alpha u_{n-1}(x) \end{pmatrix}, \quad \mathbf{w}(0) = \begin{pmatrix} \alpha(t_n) \\ 0 \end{pmatrix}. \tag{10}$$

The free boundary L_n is located wherever the representation (8) satisfies given boundary conditions on u_n and z_n. Eliminating u_n, v_n and z_n from the four expressions

$$z_n(L_n) = L_n - L_{n-1}, \quad g(u_n, u_n', L_n) = 0,$$
$$u_n = U_1 v_n + w_1, \quad z_n = U_2 v_n + w_2$$

we find that L_n must be a root of the equation

$$\phi_n(x) \equiv g\left\{\frac{U_1(x)}{U_2(x)}[x - L_{n-1} - w_2(x)] + w_1(x), \frac{1}{k}\frac{[x - L_{n-1} - w_2(x)]}{U_2(x)}, x\right\} = 0. \tag{11}$$

For example, if the moving side is held at at fixed temperature $u_n(L_n) = \beta(t_n)$ so that $g(u_n, u_n', L_n) \equiv u_n(L_n) - \beta(t_n)$ then equation (11) reduces to

$$\phi_n(x) \equiv U_2(x)[w_1(x) - \beta(t_n)] + U_1(x)[x - L_{n-1} - w_2(x)] = 0.$$

The important observation is that ϕ_n is a scalar function of x which contains only the solution $\mathbf{U}$ and $\mathbf{w}$ of well defined initial value problems for ordinary differential equations. In applications it is common to evaluate $\phi_n(x)$ at the mesh points used for the integration of (9) and (10), and to place the free boundary by linear interpolation between adjacent mesh points on which ϕ_n changes its sign. Thus, in principle, a radiation condition on the free boundary is no more difficult to handle than a prescribed temperature. In the case of multiple roots of $\phi_n(x) = 0$ the choice of the correct root must be made on the basis of the physics.

Once L_n has been determined we are faced with a standard fixed boundary problem for (5) which can be solved with a variety of methods. For the sample computation presented below the reverse sweep outlined in [4] was used to compute $u_n(x)$ on $[0, L_n]$.

It may be noted from [3] and [4] that the method is equally applicable to heat flow problems with flux or reflection conditions on the stationary boundary, to equations with convective and linear source terms and to problems where the location of the free boundary is intimately coupled to the flux and temperature conditions there (as in a Stefan or fluidized-bed coating problem). Moreover, since all coefficients in (1) may be space dependent, the method can be used for each linearized problem in the iterative solution of heat transfer problems with temperature dependent parameters.

A SAMPLE CALCULATION

In order to illustrate the performance of this algorithm we have computed the length of a bar of ingot iron with a cross section of one square inch as a function of time. It is assumed that the bar is insulated along its length, that it is initially near the melting temperature, and that the temperature at its ends is given. The following parameters were used. $L(0) = 100$ in, $u(x, 0) = 2800°$F, $k = 510$ Btu/h in °F, $c = 0.150$ Btu/lb °F, $\rho = 0.26$ lb/in^3, $\alpha = 6.5 \times 10^{-6}$ in/in °F. The bar was cooled, reheated and then cooled to 0°F according to the input

$$\alpha(t) = \beta(t) = \begin{cases} 2800\cos 2t, & t \in (0, 0.5] \\ 2800\cos 1 + 2000(t - 1.5)^2, & t \in (0.5, 1] \\ (2800\cos 1 + 500)(1.5 - t)/0.5, & t \in (1, 1.5] \\ 0, & t > 1.5. \end{cases}$$

After three hours the bar temperature was essentially zero throughout its length.

The invariant imbedding equations (9)–(11) were solved exactly as in [4]. Closed form solutions were used for U_1 and U_2; the remaining equations were integrated with the trapezoidal rule and the boundary L was placed by linear interpolation. (For more details see [4].) Below we show a computer plot of the calculated length $L(t)$ as a function of time. The computed final length is 98.194.

It is a consequence of the above model that the final length depends somewhat on the temperature history of the bar. If the cooling is uniform then it follows from (4) that the final length is given by

$$\tilde{L}(T) = L(0)\,e^{\alpha\Delta u} \qquad (12)$$

where Δu is the temperature drop. A computation of heat transfer over a period of 20 h with end temperatures $u(0,t) = u[L(t),t] = \alpha(t/10)$, where $\alpha(t)$ is the same as above, yielded uniform temperatures at all times and a final length of $L(20) = 98.19641$ while formula (12) predicts $\tilde{L}(20) = 98.19646$. On the other hand, if (4) is written in the equivalent form

$$L(T) = L(0) + \alpha \int_0^{L(T)} u(x,T)\,\mathrm{d}x - \alpha \int_0^{L(0)} u(x,0)\,\mathrm{d}x - \alpha \int_0^{L(T)} u[L(t),t]\,\frac{\partial L}{\partial t}(t)\,\mathrm{d}t$$

then it is seen that an instantaneous chilling of the free end to the final equilibrium temperature leads to a final length given by the linear expansion formula

$$\tilde{\tilde{L}}(T) = L(0)(1 + \alpha\Delta u). \qquad (13)$$

A numerical computation of heat transfer over 3 h with end temperatures of $u(0,t) = \alpha(t)$ and $u[L(t),t] = 0$ with a sequence of time steps increasing from $\Delta t = 0.0005$ to $\Delta t = 0.05$ yielded a final length of $L(3) = 98.18073$ while formula (13) predicts $\tilde{\tilde{L}}(3) = 98.1800$. Thus, the algorithm is judged to perform reliably even in the case of discontinuous boundary and initial data and to yield answers for problems with uniform initial and final temperatures which lie between the exponential formula (12) and its linear approximation (13). The algorithm, however, is in no sense tied to the existence of equilibrium temperatures.

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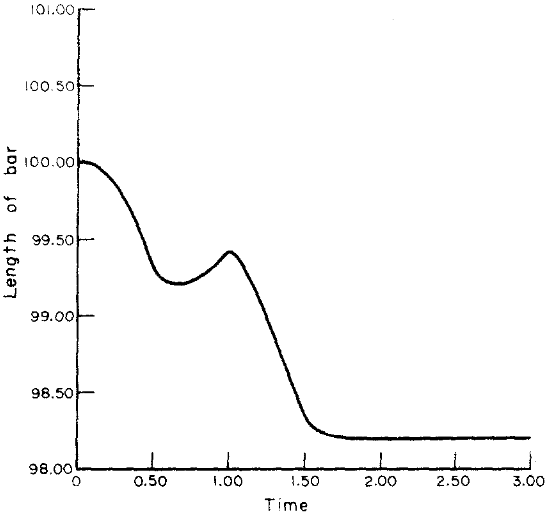

FIG. 1. Length of a bar of ingot iron as a function of time during cooling from 2800 to 0°F. $\Delta t = 3/120$ h, $\Delta x = 0.02$ on [0, 2], $\Delta x = 96/200$ on [2, 98] and $\Delta x = 2/1000$ on [98, 100]. Computing time was 44 s on the Cyber 74.

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